Discrete time approximation for continuously and discretely reflected BSDE's

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Abstract

We study the discrete time approximation of the solution (Y, Z, K) of a reflected BSDE. As in Ma and Zhang (2005), we consider a markovian setting with a reflecting barrier of the form $h(X)$ where X solves a forward SDE. We first focus on the discretely reflected case. Based on a representation for the Z component in terms of the next reflection time, we retrieve the convergence result of Ma and Zhang (2005) without their uniform ellipticity condition on X. These results are then extended to the case where the reflection operates continuously. We also improve the bound on the convergence rate when $h \in C_b^2$ with Lipschitz second derivative.

Key words: Reflected BSDEs, discrete-time approximation schemes, regularity.

MSC Classification (2000): 65C99, 60H35, 60G40.

1 Introduction

In this paper, we consider the solution (Y, Z, K) of a decoupled Forward-Backward SDE with reflection

$$
\begin{aligned}
X_t &= X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s \;, \\
Y_t &= g(X_T) + \int_t^T f(X_s, Y_s, Z_s)\mathrm{d}s - \int_t^T (Z_s)'\mathrm{d}W_s + K_T - K_t \;\;, \\
Y_t &\geq h(X_t) \;\;,\;\; t \leq T \;\text{ and }\; \int_0^t (\,Y_t - h(X_t)\,)\mathrm{d}K_t \,=\, 0\;,
\end{aligned}
$$

where b, σ, f, g and h are Lipschitz-continuous functions. Such equations appear naturally in finance in the pricing and hedging of American contingent claims, see [7]. They are more generally related to semilinear parabolic PDEs with free boundary, see [9].

We study a discrete-time approximation scheme of the form

$$
\begin{aligned}
\bar{Y}^\pi_T &= g(X^\pi_T)\;, \\
\bar{Z}^\pi_{t_i} &= (t_{i+1} - t_i)^{-1}\,\mathbb{E}_i\left[\bar{Y}^\pi_{t_{i+1}}(W_{t_{i+1}} - W_{t_i})\right] \\
\tilde{Y}^\pi_{t_i} &= \mathbb{E}_i\left[\bar{Y}^\pi_{t_{i+1}}\right] + (t_{i+1} - t_i) f(X^\pi_{t_i}, \tilde{Y}^\pi_{t_i}, \bar{Z}^\pi_{t_i}) \\
\bar{Y}^\pi_{t_i} &= \tilde{Y}^\pi_{t_i} \vee h(X^\pi_{t_i})\;,\; i \leq N-1\;,
\end{aligned}
$$

where $\pi = \{t_0 = 0 < t_1 < \ldots < t_N = T\}$ is a partition of the time interval $[0, T]$ with modulus $|\pi|$, and X^π is the Euler scheme of X.

In the non-reflected case, such approximations have been studied by [3] and [15], see also [2] and [6] for BSDEs with jumps. In all these analysis, it appears that the approximation error is intimately related to a regularity property on Z. More, precisely, the error is controlled by

$$
|\pi|^{\frac{1}{2}} + \mathbb{E}\left[\int_0^T |Z_t - \bar{Z}_t|^2 \mathrm{d}t\right]^{\frac{1}{2}}
$$

where $\bar{Z}$ is defined on $[t_i, t_{i+1})$ by $\bar{Z}_t = (t_{i+1} - t_i)^{-1}\mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z_s \mathrm{d}s \mid \mathcal{F}_{t_i}\right]$. It is shown in [14] that, in the non-reflected case, the last term is bounded by $C|\pi|^{\frac{1}{2}}$. This provides the expected rate of convergence for the discrete-time approximation scheme, see [2] for an extention to BSDEs with jumps. This result is remarkable since it does not require any ellipticity condition on σ and the coefficients are only assumed to be Lipschitz.

The reflected case is more difficult to handle except when f is independent of Z as in [1] and [3]. In this case, there is no need to control Z and the error is still bounded by $C|\pi|^{\frac{1}{2}}$. It can even be improved when h is semi-convex, see [1].

The general case was studied in [11]. When b, σ are C^1_b and h is C^2_b, they prove that $\mathbb{E}\left[\int_0^T |Z_t - \bar{Z}_t|^2 \mathrm{d}t\right]^{\frac{1}{2}}$ is bounded by $C|\pi|^{\frac{1}{4}}$. This allows to show that the discrete-time scheme converges at least at a rate $|\pi|^{\frac{1}{4}}$. Their proof relies on a particular representation of Z obtained by means of an integration by parts argument, in the Malliavin sense. It generalizes a result of [5] obtained in the non-reflected case with $f = 0$. The main drawback of this approach is that it requires some uniform ellipticity condition on σ, an assumption which was not used in the non-reflected case.

The aim of this paper is to improve this result by removing the ellipticity condition on σ. Our approach is slightly different from [11]. We first study the solution (Y^b, Z^b, K^b) of a discretely reflected BSDE. We provide a new representation result for Z^b in terms of the next reflection time. This allows us to prove that $\mathbb{E}\left[\int_0^T |Z^b_t - \bar{Z}^b_t|^2 \mathrm{d}t\right]^{\frac{1}{2}}$ is controlled by $|\pi|^{\frac{1}{4}}$ without ellipticity condition on σ. By using a standard approximation argument, we then extend this property to Z. As a consequence, we show that the discrete-time scheme approches both continuously- and discretely-reflected BSDEs at least at a rate $|\pi|^{\frac{1}{4}}$. We only assume that all the functions are Lipschitz-continuous and that h is C^1_b with Lipschitz-continuous derivatives. When h is C^2_b with Lipschitz-continuous second derivatives, this result is improved and the error on Y is shown to be bounded by $C|\pi|^{\frac{1}{2}}$ as in the non-reflected case. The error on Z is also improved when X^π coincides with X on π.

To conclude this introduction, we would like to observe that the above discrete time scheme can not be directly implemented in pratice since it requires the computation of conditional expectations. However, the numerical methods discussed in [1], [3], [4] and [6], see also the references therein, can be easily adapted to our context and do not require any further analysis.

The rest of the paper is organized as follows. In Section 2 and Section 3, we study the approximation of the discretely reflected BSDE. The representation and the regularity property of Z^b are proved in Section 5. The continuously reflected case is studied in Section 4. The Appendix contains the proof of rather standard approximation results for BSDEs.

2 The forward process

Let $T > 0$ be a finite time horizon and $(\Omega, \mathcal{F}, \mathbb{P})$ be a stochastic basis supporting a d-dimensional Brownian motion W. We assume that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$ generated by W satisfies the usual assumptions and that $\mathcal{F}_T = \mathcal{F}$.

Let X be the solution on $[0,T]$ of the stochastic differential equation

$$X_t = X_0 + \int_0^t b(X_u)\mathrm{d}u + \int_0^t \sigma(X_u)\mathrm{d}W_u$$

where $X_0 \in \mathbb{R}^d$, and, $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \mapsto \mathbb{M}^d$ are assumed to be $L-$Lipschitz, i.e.

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq L|x-y| \quad \text{for all} \quad x, y \in \mathbb{R}^d \,. \tag{2.1}$$

Here $\mathbb{M}^d$ is the space of d-dimensional matrices, $|\cdot|$ denotes the Euclidian norm on $\mathbb{R}^d$ or $\mathbb{M}^d$ and all elements of $\mathbb{R}^d$ are viewed as column vectors.
By convention, we assume that $|X_0| + T + |b(0)| + |\sigma(0)| \leq L$. In the following, we shall denote by C_L a generic positive constant which depends only on L (but may take different values). We write C_L^p if it depends from an extra parameter $p > 0$.

For later use, we recall the well-known consequence of (2.1):

$$\| \sup_{t \leq T} |X_t| \, \|_{L^p} \quad \leq \quad C_L^p \,. \tag{2.2}$$

where, for a random variable ξ, we note $\|\xi\|_{L^p} := \mathbb{E}\left[|\xi|^p\right]^{\frac{1}{p}}$.

The discrete-time approximation of X has been widely studied in the literature, see e.g. [10]. When $(X_{t_i})_{i \leq N}$ cannot be perfectly simulated, we use the standard Euler scheme X^π defined for a partition $\pi := \{0 = t_0 < t_1 < \ldots < t_N = T\}$ of $[0,T]$, $N \geq 1$, by

$$\begin{cases} X_0^\pi & = \quad X_0 \\ X_{t_{i+1}}^\pi & = \quad X_{t_i}^\pi + b(X_{t_i}^\pi)(t_{i+1} - t_i) + \sigma(X_{t_i}^\pi)(W_{t_{i+1}} - W_{t_i}) \;\; , \; i \leq N-1 \,. \end{cases}$$

In the sequel, we shall note $|\pi| := \max_{i \leq N-1}(t_{i+1} - t_i)$ the modulus of π and assume that

$$N \, |\pi| \quad \leq \quad L$$

which holds with $L \geq 1$ when the grid π is regular, i.e. $(t_{i+1} - t_i) = |\pi|$ for all $i \leq N-1$.

As usual, we define a continuous-time version of X^π by setting

$$X_t^\pi \;=\; X_{t_i}^\pi + b(X_{t_i}^\pi)(t - t_i) + \sigma(X_{t_i}^\pi)(W_t - W_{t_i}) \;\; , \; t \in [t_i, t_{i+1}) \, , \; i \leq N-1 \,. \tag{2.3}$$

Remark 2.1 It is well known that under (2.1)

$$\| \sup_{t \leq T} |X_t - X_t^\pi| \, \|_{L^p} + \max_{i < N} \| \sup_{t \in [t_i, t_{i+1}]} |X_t - X_{t_i}^\pi| \, \|_{L^p} \;\leq\; C_L^p \, |\pi|^{\frac{1}{2}} \;\; , \; p \geq 1 \,. \tag{2.4}$$

Using standards arguments one can also obtain a conditional version of this result:

$$\mathbb{E}_i\left[|X_{t_{i+1}} - X^\pi_{t_{i+1}}|^2\right] \le e^{C_L|\pi|}|X_{t_i} - X^\pi_{t_i}|^2 + C_L|\pi|^2\mathbb{E}_i\left[(X^*_T)^2\right] \quad \forall\, i \le N-1\ , \tag{2.5}$$

where $\mathbb{E}_i[\cdot]$ denotes the conditional expectation $\mathbb{E}[\cdot \mid \mathcal{F}_{t_i}]$, $i \le N$, and $X^*_T := \max_{t \le T}|X_t|$.

If nothing else is specified, X^π will always denote the above defined process. When X can be perfectly simulated at the times $(t_i)_{i \le N}$, there is no need to introduce its Euler scheme. In this case, we say that (PS) holds and define X^π as in (2.3) on $[t_i, t_{i+1})$ but with X_{t_i} in place of $X^\pi_{t_i}$, i.e.

$$X^\pi_t \;=\; X_{t_i} + b(X_{t_i})(t - t_i) + \sigma(X_{t_i})(W_t - W_{t_i})\ \ ,\ t \in [t_i, t_{i+1})\ ,\ i \le N-1\ .$$

3 Approximation scheme for discretely reflected BSDEs

3.1 Definition

In this section, we define a discretely reflected BSDE. The reflection operates only at the dates

$$0 < r_1 < \cdots < r_{\kappa-1} < T$$

for some $\kappa \ge 1$. We set $\Re = \{r_j,\ 0 \le j \le \kappa\}$ where by convention $r_0 := 0$ and $r_\kappa := T$. The solution of the discretely reflected BSDE is a couple (Y^b, Z^b) satisfying

$$Y^b_T \;=\; \tilde{Y}^b_T := g(X_T)$$

and, for $j \le \kappa - 1$ and $t \in [r_j, r_{j+1})$,

$$\left\{\begin{array}{lcl} \tilde{Y}^b_t &=& Y^b_{r_{j+1}} + \int_t^{r_{j+1}} f(\Theta^b_s)\mathrm{d}s - \int_t^{r_{j+1}} (Z^b_s)'\mathrm{d}W_s\ \ , \\ Y^b_t &=& \mathcal{R}\left(t\ ,\ X_t\ ,\ \tilde{Y}^b_t\right)\ . \end{array}\right. \tag{3.1}$$

Here $h,\ g : \mathbb{R}^d \mapsto \mathbb{R}$ satisfy $g \ge h$ on $\mathbb{R}^d$, $f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$, $\Theta^b := (X, \tilde{Y}^b, Z^b)$, $(Z^b)'$ is the transposed vector of Z^b, and

$$\mathcal{R}(t,x,y) \;\;:=\;\; y + [h(x) - y]^+\mathbf{1}_{\{t \in \Re \setminus \{0,T\}\}}\ \ ,\ \ (t,x,y) \in [0,T] \times \mathbb{R}^{d+1}\ .$$

By a solution, we mean an adapted process $(Y^b, Z^b) \in \mathcal{S}^2 \times \mathcal{H}^2$ where, for $p \ge 1$, $\mathcal{S}^p$ is the set of real valued progressively measurable U such that

$$||U||_{\mathcal{S}^p} \;:=\; \|\sup_{t \le T}|U_t|\,\|_{L^p} \;<\; \infty\ ,$$

and $\mathcal{H}^p$ is the set of progressively measurable $\mathbb{R}^d$-valued processes V satisfying

$$||V||_{\mathcal{H}^p} \;:=\; \|\left(\int_0^T |V_r|^2\mathrm{d}r\right)^{\frac{1}{2}}\|_{L^p} \;<\; \infty\ .$$

In the following, we shall extend the definition of $||\cdot||_{\mathcal{S}^p}$ and $||\cdot||_{\mathcal{H}^p}$ to processes with values in $\mathbb{R}^d$ or $\mathbb{M}^d$, these extensions being defined in a straightforward way.

Observe that the solution of (3.1) can be constructed piecewise. Assuming that g, h and f are L-Lipschitz:

$$|g(x_1)-g(x_2)|+|h(x_1)-h(x_2)|+|f(\theta_1)-f(\theta_2)| \le L\left(|x_1-x_2|+|\theta_1-\theta_2|\right)$$

for all $x_1, x_2 \in \mathbb{R}^d$ and $\theta_1, \theta_2 \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$, the existence and uniqueness of the solution follow from [13]. By convention, we assume that $|g(0)|+|h(0)|+|f(0)| \le L$.

Remark 3.1 For later use, observe that (3.1) can be written as

$$\tilde{Y}^b_t = g(X_T) + \int_t^T f(X_u, \tilde{Y}^b_u, Z^b_u)\mathrm{d}u - \int_t^T (Z^b_u)'\mathrm{d}W_u + \tilde{K}^b_T - \tilde{K}^b_t \ , \ t \le T \tag{3.2}$$

with

$$\tilde{K}^b_t \ := \ \sum_{j=1}^{\kappa-1} \left[h(X_{r_j}) - \tilde{Y}^b_{r_j}\right]^+ \mathbf{1}_{\{r_j \le t\}} \ .$$

By repeating the arguments of the proof of Proposition 3.5 in [9], we then easily check that

$$||\tilde{Y}^b||_{\mathcal{S}^2} + ||Y^b||_{\mathcal{S}^2} + ||Z^b||_{\mathcal{H}^2} + \|K^b_T\|_{L^2} \ \le \ C_L \ . \tag{3.3}$$

Recall that $C_L > 0$ is a constant independent of $\Re$.

We conclude this section with a regularity result on Y^b whose proof is given at the end of Section 5.3.

Proposition 3.1 *For all $p \ge 2$,*

$$\max_{i \le N-1} \mathbb{E}\left[\sup_{t \in (t_i, t_{i+1}]} |Y^b_{t_{i+1}} - Y^b_t|^2\right] \ \le \ C_L\ |\pi| \ .$$

3.2 Discrete-time approximation

From now on, we assume that $\Re \subset \pi$, i.e. the reflection times are included in the partition defining the Euler scheme of the forward process X.
We approximate (Y^b, Z^b) by the piecewise constant process $(\bar{Y}^\pi, \bar{Z}^\pi)$ defined by induction by

$$\left\{ \begin{array}{rcl} \bar{Z}^\pi_{t_i} & = & (t_{i+1}-t_i)^{-1}\, \mathbb{E}_i\left[\bar{Y}^\pi_{t_{i+1}}(W_{t_{i+1}} - W_{t_i})\right] \\ \tilde{Y}^\pi_{t_i} & = & \mathbb{E}_i\left[\bar{Y}^\pi_{t_{i+1}}\right] + (t_{i+1}-t_i) f(X^\pi_{t_i}, \tilde{Y}^\pi_{t_i}, \bar{Z}^\pi_{t_i}) \\ \bar{Y}^\pi_{t_i} & = & \mathcal{R}\left(t_i\ ,\ X^\pi_{t_i}\ ,\ \tilde{Y}^\pi_{t_i}\right)\ ,\ i \le N-1\ , \end{array} \right. \tag{3.4}$$

and by the terminal condition

$$\bar{Y}^\pi_T \quad = \quad \tilde{Y}^\pi_T := g(X^\pi_T)\ .$$

Recall that $\mathbb{E}_i[\cdot]$ stands for $\mathbb{E}[\cdot \mid \mathcal{F}_{t_i}]$. For ease of notations, we set

$$(\bar{Y}^\pi_t, \bar{Z}^\pi_t) = (\bar{Y}^\pi_{t_i}, \bar{Z}^\pi_{t_i}) \quad \text{for } t \in [t_i, t_{i+1}) \;\; i \leq N-1\ . \tag{3.5}$$

Using an induction argument and the Lispchitz-continuity assumption on g, h and f, one easily checks that the above processes are square integrable. It follows that the conditional expectations are well defined at each step of the algorithm.

Remark 3.2 Observe that $\tilde{Y}^\pi$ is defined implicitly as the solution of a fixed point problem. Since f is Lipschitz-continuous, it is defined with no ambiguity. Moreover, for small values of $|\pi|$ it can be estimated numerically in a very fast and accurate way, if not explicit. We refer to [2] for a discussion on the difference between implicit and explicit schemes.

For later use, let us introduce the continuous time scheme associated to $(\bar{Y}^\pi, \bar{Z}^\pi)$. By the martingale representation theorem, there exists $Z^\pi \in \mathcal{H}^2$ such that

$$\bar{Y}^\pi_{t_{i+1}} = \mathbb{E}_i\left[\bar{Y}^\pi_{t_{i+1}}\right] + \int_{t_i}^{t_{i+1}} (Z^\pi_u)' \mathrm{d}W_u \;\;,\;\; i \leq N-1\ .$$

We can then define $\tilde{Y}^\pi$ on $[t_i, t_{i+1})$ by

$$\tilde{Y}^\pi_t \quad = \quad \bar{Y}^\pi_{t_{i+1}} + (t_{i+1} - t) f(X^\pi_{t_i}, \tilde{Y}^\pi_{t_i}, \bar{Z}^\pi_{t_{i+1}}) - \int_t^{t_{i+1}} (Z^\pi_u)' \mathrm{d}W_u\ , \tag{3.6}$$

and set

$$Y^\pi_t \quad := \quad \mathcal{R}(t, X^\pi_t, \tilde{Y}^\pi_t) \;\; \text{for} \;\; t \leq T\ ,$$

so that

$$Y^\pi \;=\; \bar{Y}^\pi \text{ on } \pi \qquad \text{and} \qquad Y^\pi \;=\; \tilde{Y}^\pi \text{ on } [0,T] \setminus \Re\ . \tag{3.7}$$

Remark 3.3 It follows from the Itô isometry that

$$\bar{Z}^\pi_t \quad = \quad (t_{i+1} - t_i)^{-1} \mathbb{E}_i\left[\int_{t_i}^{t_{i+1}} Z^\pi_u \mathrm{d}u\right] \;,\; \forall\, t \in [t_i, t_{i+1})\ ,\; i \leq N-1\ , \tag{3.8}$$

recall (3.5).

3.3 Convergence results

In order to state our first result, we need to introduce the process $\bar{Z}^b$ defined on each interval $[t_i, t_{i+1})$ by

$$\bar{Z}^b_t \quad := \quad (t_{i+1} - t_i)^{-1}\mathbb{E}_i\left[\int_{t_i}^{t_{i+1}} Z^b_u \mathrm{d}u\right] . \tag{3.9}$$

Remark 3.4 For later use, observe that, by (3.8) and Jensen's inequality,

$$\mathbb{E}\left[|\bar{Z}^b_t - \bar{Z}^\pi_t|^2\right] \quad \leq \quad (t_{i+1} - t_i)^{-1}\int_{t_i}^{t_{i+1}} \mathbb{E}\left[|Z^b_u - Z^\pi_u|^2\right] \mathrm{d}u \ , \tag{3.10}$$

which implies

$$||\bar{Z}^b - \bar{Z}^\pi||_{\mathcal{H}^2} \quad \leq \quad ||Z^b - Z^\pi||_{\mathcal{H}^2} \ . \tag{3.11}$$

The following result shows that the approximation error is intimately related to the $\mathcal{H}^2$ norm of $Z^b - \bar{Z}^b$. A similar property holds in the non-reflected case, see [2], [3], [14] and [15].

Proposition 3.2 *The following holds:*

$$\max_{j\leq\kappa-1} \| \sup_{t\in[r_j,r_{j+1}]} |Y^\pi_t - Y^b_t| \, \|_{L^2} \quad \leq \quad C_L\left(|\pi|^{\frac{1}{2}} + ||Z^b - \bar{Z}^b||_{\mathcal{H}^2}\right) \ ,$$

and

$$||Z^\pi - Z^b||_{\mathcal{H}^2} \quad \leq \quad C_L\left(\kappa^{\frac{1}{2}}|\pi|^{\frac{1}{2}} + ||Z^b - \bar{Z}^b||_{\mathcal{H}^2}\right) \ .$$

If (PS) holds, then

$$||Z^\pi - Z^b||_{\mathcal{H}^2} \quad \leq \quad C_L\left(|\pi|^{\frac{1}{2}} + ||Z^b - \bar{Z}^b||_{\mathcal{H}^2}\right) \ .$$

The proof essentially follows the arguments of [3] and is provided in the appendix.

Remark 3.5 Observing that $\bar{Z}^b$ is the best $L^2(\Omega \times [0,T])$-approximation of Z^b by adapted processes which are constant on each intervalle $[t_i, t_{i+1})$, we deduce that $||Z^b - \bar{Z}^b||^2_{\mathcal{H}^2}$ goes to 0 as $|\pi|$ goes to 0. Thus, the above proposition actually shows that our discrete-time scheme is convergent. This also implies that

$$||Z^b - \bar{Z}^b||^2_{\mathcal{H}^2} \quad \leq \quad \sum_{i=0}^{N-1} \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |Z^b_t - Z^b_{t_i}|^2 \mathrm{d}t\right] \ .$$

In order to get a bound on the convergence rate, it remains to control $||Z^b - \bar{Z}^b||^2_{\mathcal{H}^2}$. Such a control will be obtained under one of the following additional assumptions.

$(\mathbf{H}1)$: $h \in C^1_b$ with L-Lipschitz derivative.

or

$(\mathbf{H}2)$: $h \in C^2_b$ with L-Lipschitz first and second derivatives.

Proposition 3.3 *Let* $(\mathbf{H}1)$ *hold. Then,*

$$||Z^b - \bar{Z}^b||_{\mathcal{H}^2} \le C_L \; \alpha(\kappa) \; |\pi|^{\frac{1}{2}} \; ,$$

where $\alpha(\kappa) = \kappa^{\frac{1}{4}}$ *under* $(\mathbf{H}1)$, *and* $\alpha(\kappa) = 1$ *under* $(\mathbf{H}2)$.

The proof will be provided in Section 5.

Combining the above propositions, we obtain the main result of this section.

Theorem 3.1 *Let* $(\mathbf{H}1)$ *hold. Then,*

$$\max_{j \le \kappa - 1} \| \sup_{t \in [r_j, r_{j+1}]} |Y^\pi_t - Y^b_t| \, \|_{L^2} \quad \le \quad C_L \; \alpha_Y(\kappa) \; |\pi|^{\frac{1}{2}}$$

and

$$||Z^\pi - Z^b||_{\mathcal{H}^2} \quad \le \quad C_L \; \alpha_Z(\kappa) \; |\pi|^{\frac{1}{2}}$$

with $(\alpha_Y(\kappa), \alpha_Z(\kappa)) = (\kappa^{\frac{1}{4}}, \kappa^{\frac{1}{2}})$ *under* $(\mathbf{H}1)$, *and* $(\alpha_Y(\kappa), \alpha_Z(\kappa)) = (1, \kappa^{\frac{1}{2}})$ *under* $(\mathbf{H}2)$.
Under (PS) the same bounds hold with $\alpha_Z(\kappa) = \kappa^{\frac{1}{4}}$ *under* $(\mathbf{H}1)$, *and* $\alpha_Z(\kappa) = 1$ *under* $(\mathbf{H}2)$.

Recalling (3.7), (3.11) and combining Proposition 3.1 with Theorem 3.1, we finally obtain a bound on the error due to the approximation of (Y^b, Z^b) by the piecewise constant process $(\bar{Y}^\pi, \bar{Z}^\pi)$ which can actually be estimated numerically, see the end of the introduction.

Corollary 3.1 *Let* $(\mathbf{H}1)$ *hold. Then,*

$$\max_{i \le N-1} \| \, |\bar{Y}^\pi_{t_i} - Y^b_{t_i}| + \sup_{t \in (t_i, t_{i+1}]} |\bar{Y}^\pi_{t_{i+1}} - Y^b_t| \, \|_{L^2} \; \le \; C_L \; \alpha_Y(\kappa) \; |\pi|^{\frac{1}{2}}$$

and

$$||\bar{Z}^\pi - Z^b||_{\mathcal{H}^2} \quad \le \quad C_L \; \alpha_Z(\kappa) \; |\pi|^{\frac{1}{2}}$$

with $(\alpha_Y(\kappa), \alpha_Z(\kappa)) = (\kappa^{\frac{1}{4}}, \kappa^{\frac{1}{2}})$ *under* $(\mathbf{H}1)$, *and* $(\alpha_Y(\kappa), \alpha_Z(\kappa)) = (1, \kappa^{\frac{1}{2}})$ *under* $(\mathbf{H}2)$.
Under (PS) the same bounds hold with $\alpha_Z(\kappa) = \kappa^{\frac{1}{4}}$ *under* $(\mathbf{H}1)$, *and* $\alpha_Z(\kappa) = 1$ *under* $(\mathbf{H}2)$.

Remark 3.6 It was shown in [11] that the results of Proposition 3.3 and Theorem 3.1 hold with the bound $C_L|\pi|^{\frac{1}{4}}$ when (Y^b, Z^b, K^b) is replaced by the solution (Y, Z, K) of a continuously reflected BSDE, see (4.1) below. Their proof is based on a particular representation of Z obtained by an integration by parts argument. However, it requires an uniform ellipticity condition on σ. Our approach is completely different. It is based on a representation for Z^b in terms of the next reflection time, see Section 5 below. This allows us to get rid of the inversibility condition on σ. The above results will be extended to the continuously reflected case in Section 4 below.

3.4 Discretely reflected BSDE constructed with the Euler scheme

In this subsection, we introduce the solution $(Y^{b,e}, Z^{b,e}, K^{b,e})$ of a discretely reflected BSDE defined as (Y^b, Z^b, K^b) but with X^π instead of X, i.e.

$$Y_T^{b,e} \quad = \quad \tilde{Y}_T^{b,e} := g(X_T^\pi)$$

and, for $j \le \kappa - 1$ and $t \in [r_j, r_{j+1})$,

$$\left\{ \begin{array}{lll} \tilde{Y}_t^{b,e} & = & Y_{r_{j+1}}^{b,e} + \int_t^{r_{j+1}} f(\Theta_u^{b,e}) \mathrm{d}s - \int_t^{r_{j+1}} (Z_s^{b,e})' \mathrm{d}W_s \ , \\ Y_t^{b,e} & = & \mathcal{R}\left(t \ , \ X_t^\pi \ , \ \tilde{Y}_t^{b,e}\right) \ . \end{array} \right. \tag{3.12}$$

with $\Theta^{b,e} := (X^\pi, \tilde{Y}^{b,e}, Z^{b,e})$.
This construction will be useful to extend the results of the previous section to the continuously reflected case.

Observe that

$$\tilde{Y}_t^{b,e} = g(X_T^\pi) + \int_t^T f(\Theta^{b,e}) \mathrm{d}u - \int_t^T (Z_u^{b,e})' \mathrm{d}W_u + \tilde{K}_T^{b,e} - \tilde{K}_t^{b,e} \ \ , \ t \le T \ ,$$

with

$$\tilde{K}_t^{b,e} \quad := \quad \sum_{j=1}^{\kappa-1} \left[h(X_{r_j}^\pi) - \tilde{Y}_{r_j}^{b,e} \right]^+ \mathbf{1}_{r_j \le t} \ .$$

Moreover, it follows from the same arguments as in the proof of Proposition 3.2, see step 5 of the proof in the Appendix, that

$$||Z^\pi - Z^{b,e}||_{\mathcal{H}^2} \quad \le \quad C_L \left(|\pi|^{\frac{1}{2}} + ||Z^{b,e} - \bar{Z}^{b,e}||_{\mathcal{H}^2} \right) \ , \tag{3.13}$$

where $\bar{Z}^{b,e}$ is defined similarly as $\bar{Z}^b$, i.e.

$$\bar{Z}_t^{b,e} \quad := \quad (t_{i+1} - t_i)^{-1} \, \mathbb{E}_i \left[\int_{t_i}^{t_{i+1}} Z_s^{b,e} \mathrm{d}s \right] \ \ , \ \ t \in [t_i, t_{i+1}), \ i \le N-1 \ .$$

We shall also prove in Section 5 that the result of Proposition 3.3 can be extended to $Z^{b,e}$.

Proposition 3.4 *Let* (**H**1) *hold. Then,*

$$||Z^{b,e} - \bar{Z}^{b,e}||_{\mathcal{H}^2} \le C_L \ \left(\kappa^{\frac{1}{4}} \ |\pi|^{\frac{1}{2}} + |\pi|^{\frac{1}{4}} \right) \ .$$

4 Extension to the continuously reflected case

Let (Y, Z, K) be the $\mathbb{F}$-progressively measurable process satisfying

$$\begin{aligned} Y_t &= g(X_T) + \int_t^T f(X_s, Y_s, Z_s)\mathrm{d}s - \int_t^T (Z_s)'\mathrm{d}W_s + K_T - K_t \ , \\ Y_t &\geq h(X_t) \ , \ 0 \leq t \leq T \end{aligned} \tag{4.1}$$

with K continuous, non-decreasing, such that $K_0 = 0$ and

$$\int_0^T (Y_t - h(X_t))\mathrm{d}K_t = 0 \ . \tag{4.2}$$

Existence and uniqueness of a solution $(Y, Z, K) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{S}^2$ follows from Theorem 5.2 in [9], recall that g, h and f are Lipschitz-continuous.

As in Section 3.4, we also define (Y^e, Z^e, K^e) as the solution of (4.1) with X^π in place of X, i.e.

$$\begin{aligned} Y_t^e &= g(X_T^\pi) + \int_t^T f(X_s^\pi, Y_s^e, Z_s^e)\mathrm{d}s - \int_t^T (Z_s^e)'\mathrm{d}W_s + K_T^e - K_t^e \ , \\ Y_t^e &\geq h(X_t^\pi) \ , \ 0 \leq t \leq T \ , \end{aligned}$$

where K^e is continuous and non-decreasing, $K_0^e = 0$ and $\int_0^T (Y_t^e - h(X_t^\pi))\mathrm{d}K_t^e = 0$.

Our first result is rather standard. It shows that (Y, Z) and (Y^e, Z^e) can be approximated by the solutions of discretely reflected BSDEs at a speed $|\Re|^{\frac{1}{2}}$ under the assumption:

$(\mathbf{H}3)$: There exists $\rho_1 : \ \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\rho_2 : \ \mathbb{R}^d \mapsto \mathbb{R}_+$ such that

$$\begin{aligned} |\rho_1(x)| + |\rho_2(x)| &\leq C_L(1 + |x|^{C_L}) \\ h(x) - h(y) &\leq \rho_1(x)'(y - x) + \rho_2(x)|x - y|^2 \ , \ \forall \, x, y \in \mathbb{R}^d \ . \end{aligned}$$

Remark 4.1 This condition is slightly weaker than the semi-convexity assumption of Definition 1 in [1] which is satisfied whenever $(\mathbf{H}1)$ or $(\mathbf{H}2)$ hold.

Proposition 4.1 *Assume that* $(\mathbf{H}3)$ *holds. Then,*

$$\sup_{t \in [0,T]} \|Y_t - Y_t^b\|_{L^2} + ||Z - Z^b||_{\mathcal{H}^2} \leq C_L \, |\Re|^{\frac{1}{2}}$$

and

$$\sup_{t \in [0,T]} \|Y_t^e - Y_t^{b,e}\|_{L^2} + ||Z^e - Z^{b,e}||_{\mathcal{H}^2} \leq C_L \, |\Re|^{\frac{1}{2}} \ .$$

If moreover $(\mathbf{H}1)$ *holds, then*

$$\max_{j \leq \kappa - 1} \left(\| \sup_{t \in [r_j, r_{j+1}]} |Y_t - Y_t^b| \, \|_{L^2} + \| \sup_{t \in [r_j, r_{j+1}]} |Y_t^e - Y_t^{b,e}| \, \|_{L^2} \right) \leq C_L \, |\Re|^{\frac{1}{2}} \ .$$

The proof is provided in the Appendix.

We can now extend the convergence results of the previous section to the continuously reflected case.

Theorem 4.1 *Let* ($\mathbf{H}1$) *hold, then*

$$\max_{i\le N-1} \| \sup_{t\in(t_i,t_{i+1}]} |\bar{Y}^{\pi}_{t_{i+1}} - Y_t| + \sup_{t\in[t_i,t_{i+1}]} |Y^{\pi}_t - Y_t| \,\|_{L^2} \;\le\; C_L\, \alpha(\pi)$$
$$\text{and } ||\bar{Z}^{\pi} - Z||_{\mathcal{H}^2} + ||Z^{\pi} - Z||_{\mathcal{H}^2} \;\le\; C_L\, |\pi|^{\frac14} \,,$$

with $\alpha(\pi) = |\pi|^{\frac14}$ *under* ($\mathbf{H}1$) *and* $\alpha(\pi) = |\pi|^{\frac12}$ *under* ($\mathbf{H}2$).
If ($\mathbf{H}2$) *and (PS) hold, then*

$$||\bar{Z}^{\pi} - Z||_{\mathcal{H}^2} + ||Z^{\pi} - Z||_{\mathcal{H}^2} \;\le\; C_L\, |\pi|^{\frac12} \,,$$

Proof. 1.The error on Y and the estimate on Z under ($\mathbf{H}2$) and (PS) follow from Proposition 4.1, Corollary 3.1 and Theorem 3.1 applied with $\Re = \pi$.
2. The estimate for Z in the general case is a bit more involved. We first approximate (Y,Z) by (Y^e,Z^e). It follows from Proposition 3.6 in [9], our Lipschitz-continuity assumptions, (2.2) and (2.4) that $||Z - Z^e||^2_{\mathcal{H}^2} \le C_L\sqrt{|\pi|}$. Then, we approximate (Y^e,Z^e) by $(Y^{b,e},Z^{b,e})$ defined in Section 3.3. By Proposition 4.1, $||Z^e - Z^{b,e}||^2_{\mathcal{H}^2} \le C_L\ |\pi|$. Finally, it follows from (3.13) that $||Z^{\pi} - Z^{b,e}||^2_{\mathcal{H}^2} \le C_L\left(|\pi| + ||Z^{b,e} - \bar{Z}^{b,e}||^2_{\mathcal{H}^2}\right)$, where the last term is controlled by Proposition 3.4. To conclude, we deduce from Jensen's inequality that $||\bar{Z}^{\pi} - Z^{b,e}||_{\mathcal{H}^2} \le ||Z^{\pi} - Z^{b,e}||_{\mathcal{H}^2} + ||Z^{b,e} - \bar{Z}^{b,e}||_{\mathcal{H}^2}$, recall (3.8). □

As in (3.9), we now define

$$\begin{aligned}
\bar{Z}_t &:= (t_{i+1}-t_i)^{-1}\mathbb{E}_i\left[\int_{t_i}^{t_{i+1}} Z_u \mathrm{d}u\right] \,, \\
\bar{Z}^e_t &:= (t_{i+1}-t_i)^{-1}\mathbb{E}_i\left[\int_{t_i}^{t_{i+1}} Z^e_u \mathrm{d}u\right] \quad \text{for } \ t\in[t_i,t_{i+1}) \,,\ i\le N-1 \,.
\end{aligned}$$

Observe that, by Jensen's inequality,

$$||\bar{Z}^b - \bar{Z}||_{\mathcal{H}^2} \;\le\; ||Z^b - Z||_{\mathcal{H}^2} \ \text{ and } \ ||\bar{Z}^{b,e} - \bar{Z}^e||_{\mathcal{H}^2} \;\le\; ||Z^{b,e} - Z^e||_{\mathcal{H}^2} \,. \quad (4.3)$$

Combining (4.3), Proposition 4.1, Proposition 3.3 and Proposition 3.4 for $\Re = \pi$, we obtain the following regularity result for Z and Z^e.

Corollary 4.1 *Let* ($\mathbf{H}1$) *holds, then*

$$||Z - \bar{Z}||_{\mathcal{H}^2} + ||Z^e - \bar{Z}^e||_{\mathcal{H}^2} \le C_L\, |\pi|^{\frac14} \,.$$

If moreover ($\mathbf{H}2$) *holds, then*

$$||Z - \bar{Z}||_{\mathcal{H}^2} \le C_L\, |\pi|^{\frac12} \,.$$

Remark 4.2 As explained in the previous section, similar results were obtained in [11]. However, their approach requires that σ is uniformly elliptic. Here, we do not need this condition on σ. We also obtain better bounds for $||Z-\bar{Z}||_{\mathcal{H}^2}$ and $\sup_{t\in[0,T]}\|Y_t^\pi - Y_t\|_{L^2}$ under (**H**2). This last assumption is slightly stronger than the C_b^2 regularity imposed on h by [11].

5 Representation and regularity of Z^b and $Z^{b,e}$

5.1 Preliminaries

In the sequel, we denote by $\mathbb{D}^{1,2}$ the space of random variable F which are differentiable in the Malliavin sense and such that

$$\|F\|_{L^2}^2 + \int_0^T \|D_tF\|_{L^2}^2 \mathrm{d}t < \infty\ .$$

Here, D_tF denotes the Malliavin derivative of F at time $t \le T$, see e.g. [12].
We also introduce the space $\mathbb{L}^{1,2}$ of adapted processes V such that, after possibly passing to a suitable version, $V_s \in \mathbb{D}^{1,2}$ for all $s \le T$ and

$$||V||_{\mathcal{H}^2} + \int_0^T ||D_tV||_{\mathcal{H}^2}\mathrm{d}t < \infty\ .$$

In the following, we shall always consider a suitable version if necessary.

In this section, we work under the stronger assumptions:

($\mathbf{H}'$): b, σ, g and f are C_b^1.

The general case will be obtained by using an approximation argument.

Remark 5.1 It is well known that under the above assumptions $X \in \mathbb{L}^{1,2}$, see e.g. [12], and satisfies for $p \ge 2$

$$\sup_{s\le T}\|D_sX_t - D_sX_u\|_{L^p} + ||D_tX - D_uX||_{\mathcal{S}^p} \quad\le\quad C_L^p\,|t-u|^{\frac{1}{2}}\ \ ,\ t,u\le T\ . \quad (5.1)$$

Moreover, the first variation process ∇X of X is well defined and solves on $[0,T]$

$$\nabla X_t \quad=\quad I_d + \int_0^t \nabla b(X_r)\nabla X_r \mathrm{d}r + \int_0^t \sum_{j=1}^d \nabla\sigma^j(X_r)\nabla X_r \mathrm{d}W_r^j$$

where I_d is the identity matrix of $\mathbb{M}^d$, σ^j is the j-th column of σ, and ∇b, $\nabla\sigma_j$ the Jacobian matrix of b and σ_j. Its inverse $(\nabla X)^{-1}$ is the solution on $[0,T]$ of

$$\begin{aligned}(\nabla X)_t^{-1} \quad&=\quad I_d - \int_0^t (\nabla X)_r^{-1}\left[\nabla b(X_r) - \sum_{j=1}^d \nabla\sigma^j(X_r)\nabla\sigma^j(X_r)\right]\mathrm{d}r \\ &\qquad - \int_0^t \sum_{j=1}^d (\nabla X)_r^{-1}\nabla\sigma^j(X_r)\mathrm{d}W_r^j\ ,\end{aligned}$$

and the following standard estimates hold:

$$||\nabla X||_{\mathcal{S}^p} + ||(\nabla X)^{-1}||_{\mathcal{S}^p} \quad \leq \quad C_L^p \ . \tag{5.2}$$

Finally, we recall the well-known relation between ∇X and DX:

$$D_t X_s \quad = \quad \nabla X_s (\nabla X_t)^{-1} \sigma(X_t) \mathbf{1}_{t \leq s} \quad \text{for all } t, s \leq T \ . \tag{5.3}$$

Using the above estimates, (2.2) and the Lipschitz-continuity of σ, we deduce that

$$|| \sup_{s \leq T} |D_s X| \, ||_{\mathcal{S}^p} \quad \leq \quad C_L^p \ . \tag{5.4}$$

Remark 5.2 Observe that X^π also belongs to $\mathbb{L}^{1,2}$ under $(\mathbf{H}')$ and satisfies

$$D_s X_t^\pi \quad = \quad \sigma(X_{\phi_s}^\pi) + \int_s^t \nabla b(X_{\phi_r}^\pi) D_s X_{\phi_r}^\pi \mathrm{d}r + \int_s^t \sum_{j=1}^d \nabla \sigma^j (X_{\phi_r}^\pi) D_s X_{\phi_r}^\pi \mathrm{d}W_r^j$$

where $\phi_t = \max\{u \in \pi \ : \ u \leq t\}$. Thus, $D_s X_t^\pi$ is given by

$$\left\{ \prod_{k \in N_{s,t}} \left(I_d + \nabla b(X_{t_k}^\pi)(t_{k+1} \wedge t - t_k) + \sum_{j=1}^d \nabla \sigma^j (X_{t_k}^\pi)(W_{t_{k+1} \wedge t}^j - W_{t_k}^j) \right) \right\} \sigma(X_{\phi_s}^\pi)$$

with $N_{s,t} := \{k \leq N \ : \ s \leq t_k < t\}$. Using the bound on ∇b and $\nabla \sigma^j$, $j \leq d$, we obtain

$$\mathbb{E}\left[\sup_{s,t \leq T} |D_s X_t^\pi|^p \right] \quad \leq \quad C_L^p \left(1 + C_L^p |\pi|^{2p}\right)^N \left(1 + \mathbb{E}\left[\sup_{t \leq T} |X_t^\pi|^{2p} \right] \right)^{\frac{1}{2}}$$

which leads to

$$\mathbb{E}\left[\sup_{s \leq t \leq T} |D_s X_t^\pi|^p \right] \quad \leq \quad C_L^p \ , \ p \geq 1 \ . \tag{5.5}$$

By using standard arguments, one also easily checks that the bounds (5.1) can be extended to X^π, uniformly in π:

$$\sup_{s \leq T} \|D_s X_t^\pi - D_s X_u^\pi\|_{L^p} + ||D_t X^\pi - D_u X^\pi||_{\mathcal{S}^p} \ \leq \ C_L^p \ |t-u|^{\frac{1}{2}} \ \ , \ t, u \leq T \ . \tag{5.6}$$

5.2 Representation

In order to provide a suitable representation of Z^b, we shall appeal to the following easy lemma.

Lemma 5.1 *If $F \in \mathbb{D}^{1,2}$, then $[F]^+ \in \mathbb{D}^{1,2}$ and $D_t [F]^+ = (D_t F) \mathbf{1}_{\{F > 0\}}$.*

Proof. By a straightforward adaptation of Proposition 1.2.3 in [12], we observe that $[F]^+$ belongs to $\mathbb{D}^{1,2}$ and $D_t[F]^+ = \alpha(D_t F)$ where α is a random variable bounded by 1 satisfying $\mathbf{1}_{\{F>0\}}\alpha = \mathbf{1}_{\{F>0\}}$. The proof is then concluded by appealing to Proposition 1.3.7 in [12]. □

Recalling that $g \geq h$, using Remark 5.1, Lemma 5.1, Proposition 5.3 in [8] and an induction argument, we easily deduce from (3.1) that $(\tilde{Y}^b, Z^b)$ belongs to $\mathbb{L}^{1,2}$.

Proposition 5.1 *Let* $(\mathbf{H}')$ *hold. Then, the process* $(\tilde{Y}^b, Z^b)$ *belongs to* $\mathbb{L}^{1,2}$ *and, for all* $t \leq T$, $D_t(\tilde{Y}^b, Z^b)$ *solves on* $[r_j, r_{j+1})$, $j \leq \kappa - 1$,

$$\begin{aligned} D_t\tilde{Y}^b_s &= (D_t h(X_{r_{j+1}}) - D_t\tilde{Y}^b_{r_{j+1}})\mathbf{1}_{\{h(X_{r_{j+1}})>\tilde{Y}^b_{r_{j+1}}\}} \\ &+ D_t\tilde{Y}^b_{r_{j+1}} + \int_s^{r_{j+1}} \nabla f(\Theta^b_u)D_t\Theta^b_u \mathrm{d}u - \int_s^{r_{j+1}} D_t Z^b_s \mathrm{d}W_s \ . \end{aligned} \tag{5.7}$$

In order to get rid of the indicator functions appearing in (5.7), we now define the following sequence of stopping times

$$\tau_j = \inf\{t \in \Re \mid t \geq r_{j+1},\ h(X_t) > \tilde{Y}^b_t\} \wedge T \ \ ,\ j \leq \kappa - 1 \ .$$

Following [14] , we also define

$$\Lambda^s_t \ := \ \exp\left\{\int_s^t \nabla_z f(\Theta^b_u)'\mathrm{d}W_u - \int_s^t \left(\frac{1}{2}|\nabla_z f(\Theta^b_u)|^2 + \nabla_y f(\Theta^b_u)\right)\mathrm{d}u\right\} \ ,\ s \leq t \leq T \ ,$$

where $\nabla_y f$ denote the partial derivative of f with respect to its second variable y, and $(\nabla_x f)'$ and $(\nabla_z f)'$ the gradient of f with respect to its first and last variable.

Remark 5.3 The following estimates are standard:

$$\|\sup_{s\leq t\leq T} \Lambda^s_t\|_{L^p} \ \leq \ C^p_L \ , \tag{5.8}$$

$$\|\sup_{u\leq t\wedge s} |\Lambda^u_t - \Lambda^u_s|\ \|_{L^p} \ \leq \ C^p_L|t-s|^{\frac{1}{2}} \ ,\ t,s \leq T \ . \tag{5.9}$$

Using (5.1), we deduce that

$$||\sup_{u\vee t\leq s\leq T} |\Lambda^t_s D_t X_s - \Lambda^u_s D_u X_s|\ ||_{\mathcal{S}^p} \ \leq \ C^p_L\ |t-u|^{\frac{1}{2}} \ ,\ u,t \leq T \ . \tag{5.10}$$

We can now state the main result of this section which provides a representation for Z^b.

Corollary 5.1 *Let* $(\mathbf{H}')$ *hold. Then, there is a version of* Z^b *such that for each* $j \leq \kappa - 1$ *and* $t \in [r_j, r_{j+1})$*:*

$$\begin{aligned} (Z^b_t)' &= \mathbb{E}\left[\nabla g(X_T)(\Lambda^t D_t X)_T\mathbf{1}_{\{\tau_j=T\}} + \nabla h(X_{\tau_j})(\Lambda^t D_t X)_{\tau_j}\mathbf{1}_{\{\tau_j<T\}} \mid \mathcal{F}_t\right] \\ &+ \mathbb{E}\left[\int_t^{\tau_j} \nabla_x f(\Theta^b_u)(\Lambda^t D_t X)_u \mathrm{d}u \mid \mathcal{F}_t\right] \ . \end{aligned}$$

Proof. 1. It follows from Proposition 5.1 and the assumption $g \geq h$ that, for all $t \leq T$, $j \leq \kappa - 1$ and $s \in [r_j, r_{j+1})$, we have

$$
\begin{aligned}
D_t \tilde{Y}^b_s &= \left(\nabla h(X_{r_{j+1}}) D_t X_{r_{j+1}} - D_t \tilde{Y}^b_{r_{j+1}}\right) \mathbf{1}_{\{h(X_{r_{j+1}}) > \tilde{Y}^b_{r_{j+1}}\}} \\
&+ D_t \tilde{Y}^b_{r_{j+1}} + \int_s^{r_{j+1}} \nabla f(\Theta^b_u) D_t \Theta^b_u \mathrm{d}u - \int_s^{r_{j+1}} D_t Z^b_u \mathrm{d}W_u \, .
\end{aligned}
$$

In particular,

$$
\begin{aligned}
D_t \tilde{Y}^b_{r_j} &= \left(\nabla h(X_{r_{j+1}}) D_t X_{r_{j+1}} - D_t \tilde{Y}^b_{r_{j+1}}\right) \mathbf{1}_{\{h(X_{r_{j+1}}) > \tilde{Y}^b_{r_{j+1}}\}} \\
&+ D_t \tilde{Y}^b_{r_{j+1}} + \int_{r_j}^{r_{j+1}} \nabla f(\Theta^b_u) D_t \Theta^b_u \mathrm{d}u - \int_{r_j}^{r_{j+1}} D_t Z^b_u \mathrm{d}W_u \, .
\end{aligned}
$$

Since $\tilde{Y}^b_{r_\kappa} = g(X_T)$, it follows that $D_t \tilde{Y}^b_{r_\kappa} = \nabla g(X_T) D_t X_T$. Recalling that $g \geq h$, it then results from a simple induction that for $s \in [r_j, r_{j+1})$

$$
\begin{aligned}
D_t \tilde{Y}^b_s &= \nabla g(X_T) D_t X_T \mathbf{1}_{\{\tau_j = T\}} + \nabla h(X_{\tau_j})(D_t X)_{\tau_j} \mathbf{1}_{\{\tau_j < T\}} \\
&+ \int_s^{\tau_j} \nabla f(\Theta^b_u) D_t \Theta^b_u \mathrm{d}u - \int_s^{\tau_j} D_t Z^b_s \mathrm{d}W_s \, .
\end{aligned}
$$

By the same arguments as in Proposition 5.3 in [8], we have $D_t \tilde{Y}^b_t = D_t Y^b_t = (Z^b_t)'$ on (r_j, r_{j+1}). The result then follows from the previous equation, Itô's formula and by considering a suitable version. □

Remark 5.4 Assume that $(\mathbf{H}')$ holds. Then, it follows from (5.4), (5.8) and Corollary 5.1 that $||Z^b||_{\mathcal{S}^p} \leq C^p_L$.

Remark 5.5 Let $(\mathbf{H}')$ hold. We deduce from the same arguments as in the proof of Corollary 5.1 that there is a version of $Z^{b,e}$ such that for each $t \in [r_j, r_{j+1})$, $j \leq \kappa - 1$:

$$
\begin{aligned}
(Z^{b,e}_t)' &= \mathbb{E}\left[\nabla g(X_T) D_t X^\pi_T \mathbf{1}_{\{\tau^e_j = T\}} + \nabla h(X^\pi_{\tau^e_j})(\Lambda^{e,t} D_t X^\pi)_{\tau_j} \mathbf{1}_{\{\tau^e_j < T\}} \mid \mathcal{F}_t\right] \\
&+ \mathbb{E}\left[\int_t^{\tau^e_j} \nabla_x f(\Theta^{b,e}_u)(\Lambda^{e,t} D_t X^\pi)_u \mathrm{d}u \mid \mathcal{F}_t\right] \, , \; t \leq T \, ,
\end{aligned}
$$

where

$$
\tau^e_j = \inf\{t \in \Re \mid t \geq r_{j+1}, \; h(X^\pi_t) > \tilde{Y}^{b,e}_t\} \wedge T \;\; , \; j \leq \kappa - 1 \, .
$$

and $\Lambda^{e,s}_t$ is defined, for $s \leq t \leq T$, by

$$
\Lambda^{e,s}_t := \exp\left\{\int_s^t \nabla_z f(\Theta^{b,e}_u)' \mathrm{d}W_u - \int_s^t \left(\frac{1}{2}|\nabla_z f(\Theta^{b,e}_u)|^2 + \nabla_y f(\Theta^{b,e}_u)\right) \mathrm{d}u\right\} \, .
$$

The following estimates are standard:

$$
\| \sup_{s \leq t \leq T} \Lambda^{e,s}_t \|_{L^p} \leq C^p_L \, , \tag{5.11}
$$

$$
\| \sup_{u \leq t \wedge s} |\Lambda^{e,u}_t - \Lambda^{e,u}_s| \, \|_{L^p} \leq C^p_L |t - s|^{\frac{1}{2}} \;\; , \; t, s \leq T \, . \tag{5.12}
$$

Using (5.6), we deduce that

$$\| \sup_{t\vee u\le s\le T} |\Lambda_s^{e,t} D_t X_s^\pi - \Lambda_s^{e,u} D_u X_s^\pi| \, \|_{L^p} \quad \le \quad C_L^p \, |t-u|^{\frac{1}{2}} \ \ , \ u,t \le T \ . \qquad (5.13)$$

5.3 Regularity

In this section, we replace $(\mathbf{H}2)$ by the stronger assumption:

$(\mathbf{H}2')$: $h \in C_b^3$, with derivatives up to order three bounded by L.

The extension of the following results to $(\mathbf{H}2)$ will be obtained by using an approximation argument.

Proposition 5.2 *Let* $(\mathbf{H}1)$ *hold. Then*

$$||Z^b - \bar{Z}^b||_{\mathcal{H}^2} \le C_L \ \alpha(\kappa) \ |\pi|^{\frac{1}{2}} \ ,$$

where $\alpha(\kappa) = \kappa^{\frac{1}{4}}$ *under* $(\mathbf{H}1)$*, and* $\alpha(\kappa) = 1$ *under* $(\mathbf{H}2')$.

The following remark prepares for the proof.

Remark 5.6 Set

$$\beta \quad := \quad \left(1 + \sup_{s\le t\le T} |D_s X_t| + \sup_{t\le T} |X_t| + \sup_{s\le t\le T} |\Lambda_t^s| \right)^4 \quad ,$$

and observe that, by (2.2), (5.4) and (5.8),

$$||\beta||_{\mathcal{S}^p} \quad \le \quad C_L^p \ , \ p \ge 2 \ . \qquad (5.14)$$

Fix $t \le T$ and let τ_1 and τ_2 be two stopping times such that $t \le \tau_1 \le \tau_2 \le T$ $\mathbb{P}-$a.s. By the Lipschitz-continuity assumption on b and σ, we have

$$\mathbb{E}\left[|X_{\tau_1} - X_{\tau_2}|^2 \mid \mathcal{F}_{\tau_1} \right] \quad \le \quad C_L \, \mathbb{E}\left[\beta(\tau_2 - \tau_1) \mid \mathcal{F}_{\tau_1} \right] \ . \qquad (5.15)$$

Under $(\mathbf{H}2')$, we deduce from Itô's Lemma that

$$\left| \mathbb{E}\left[\nabla h(X_{\tau_2}) \Lambda_{\tau_2}^t (D_t X)_{\tau_2} - \nabla h(X_{\tau_1}) \Lambda_{\tau_1}^t (D_t X)_{\tau_1} \mid \mathcal{F}_{\tau_1} \right] \right| \le C_L \, \mathbb{E}\left[\beta(\tau_2 - \tau_1) \mid \mathcal{F}_{\tau_1} \right] \ . \qquad (5.16)$$

When $(\mathbf{H}1)$ holds, we can use the bound $|\nabla h| \le L$ to obtain

$$\begin{aligned} \left| \nabla h(X_{\tau_2}) \Lambda_{\tau_2}^t (D_t X)_{\tau_2} - \nabla h(X_{\tau_1}) \Lambda_{\tau_1}^t (D_t X)_{\tau_1} \right| \quad &\le \quad \beta \, |\nabla h(X_{\tau_2}) - \nabla h(X_{\tau_1})| \\ &+ \quad C_L \left| \Lambda_{\tau_2}^t (D_t X)_{\tau_2} - \Lambda_{\tau_1}^t (D_t X)_{\tau_1} \right| \ , \end{aligned}$$

which, by Lipschitz-continuity of ∇h, Itô's Lemma and the Cauchy-Schwartz inequality, implies

$$\mathbb{E}\left[\left| \nabla h(X_{\tau_2}) \Lambda_{\tau_2}^t (D_t X)_{\tau_2} - \nabla h(X_{\tau_1}) \Lambda_{\tau_1}^t (D_t X)_{\tau_1} \right| \mid \mathcal{F}_{\tau_1} \right] \le C_L \left(\bar{\beta} \, \mathbb{E}\left[\beta(\tau_2 - \tau_1) \mid \mathcal{F}_{\tau_1} \right] \right)^{\frac{1}{2}} \qquad (5.17)$$

where

$$\bar{\beta} := \sup_{t \le T} \mathbb{E}\left[\beta^2 \mid \mathcal{F}_t\right] \quad \text{satisfies} \quad ||\bar{\beta}||_{\mathcal{S}^p} \le C_L^p \ , \ p \ge 2 \ , \tag{5.18}$$

recall (5.14).

Proof of Proposition 5.2.

1. It follows from Corollary 5.1 that, after passing to a suitable version,

$$(Z_t^b)' = V_t^{j,t} \quad , \ r_j \le t < r_{j+1} \ , \ j \le \kappa - 1 \ ,$$

where, for $j \le \kappa - 1$,

$$\begin{aligned} V_t^{j,s} \ &:= \ \mathbb{E}\left[\nabla g(X_T)(\Lambda^s D_s X)_T \mathbf{1}_{\{\tau_j = T\}} + \nabla h(X_{\tau_j})(\Lambda^s D_s X)_{\tau_j} \mathbf{1}_{\{\tau_j < T\}} \mid \mathcal{F}_t\right] \\ &+ \ \mathbb{E}\left[\int_s^{\tau_j} \nabla_x f(\Theta_u^b)(\Lambda^s D_s X)_u \mathrm{d}u \mid \mathcal{F}_t\right] \ , \quad s \le t \ . \end{aligned}$$

For $t \in [t_i, t_{i+1}) \subset [r_j, r_{j+1})$, we then have

$$|Z_t^b - Z_{t_i}^b| \le |V_t^{j,t} - V_t^{j,t_i}| + |V_t^{j,t_i} - V_{t_i}^{j,t_i}| \ , \tag{5.19}$$

where, by (5.10),

$$\|V_t^{j,t} - V_t^{j,t_i}\|_{L^2}^2 \quad \le \quad C_L \ |\pi| \ . \tag{5.20}$$

2. Using the martingale property of V^{j,t_i} on $[t_i, t_{i+1}]$ and (5.3), we deduce that

$$\mathbb{E}\left[|V_t^{j,t_i} - V_{t_i}^{j,t_i}|^2\right] \quad \le \quad \mathbb{E}\left[|V_{t_{i+1}}^{j,t_i}|^2 - |V_{t_i}^{j,t_i}|^2\right] \ \le \ \mathbb{E}\left[(|A_{t_{i+1}}^j|^2 - |A_{t_i}^j|^2)|\eta_{t_i}|^2\right]$$

where

$$\begin{aligned} A_t^j \ &:= \ \mathbb{E}\left[\nabla g(X_T)\Lambda_T^0 \nabla X_T \mathbf{1}_{\{\tau_j = T\}} + \nabla h(X_{\tau_j})(\Lambda^0 \nabla X)_{\tau_j} \mathbf{1}_{\{\tau_j < T\}} \mid \mathcal{F}_t\right] \\ &+ \mathbb{E}\left[\int_t^{\tau_j} \nabla_x f(\Theta_u^b)(\Lambda^0 \nabla X)_u \mathrm{d}u \mid \mathcal{F}_t\right] \quad , \quad t \le T \end{aligned}$$

and

$$\eta_t \quad := \quad (\Lambda_t^0 \nabla X_t)^{-1} \sigma(X_t) \ , \ \ t \le T \ .$$

Using the Lipschitz-continuity of σ, we observe that

$$\mathbb{E}\left[|\eta_{t_{i+1}} - \eta_{t_i}|^4\right]^{\frac{1}{2}} \quad \le \quad C_L \ |\pi| \ .$$

By (2.2), (5.2), (5.3), (5.8) and Cauchy-Schwartz inequality, it then follows that

$$\begin{aligned} \mathbb{E}\left[|V_t^{j,t_i} - V_{t_i}^{j,t_i}|^2\right] \quad &\le \quad \mathbb{E}\left[|A_{t_{i+1}}^j \eta_{t_{i+1}}|^2 - |A_{t_i}^j \eta_{t_i}|^2 + |A_{t_{i+1}}^j|^2 |\eta_{t_{i+1}} - \eta_{t_i}|^2\right] \\ &\le \quad \mathbb{E}\left[|V_{t_{i+1}}^{j,t_{i+1}}|^2 - |V_{t_i}^{j,t_i}|^2\right] + C_L \ |\pi| \ . \end{aligned} \tag{5.21}$$

3. It remains to study the first term in the right-hand side of (5.21). Define i_j through $t_{i_j} = r_j$, $j \le \kappa$ and observe that

$$
\begin{aligned}
\Sigma &:= \sum_{j=0}^{\kappa-1} \sum_{k=i_j}^{i_{j+1}-1} \mathbb{E}\left[|V^{j,t_{k+1}}_{t_{k+1}}|^2 - |V^{j,t_k}_{t_k}|^2\right] \\
&= \sum_{j=0}^{\kappa-1} \mathbb{E}\left[|V^{j,r_{j+1}}_{r_{j+1}}|^2 - |V^{j,r_j}_{r_j}|^2\right] \\
&\le \mathbb{E}\left[|V^{\kappa-1,r_\kappa}_{r_\kappa}|^2 - |V^{0,r_0}_{r_0}|^2\right] + \sum_{j=1}^{\kappa-1} \mathbb{E}\left[|V^{j-1,r_j}_{r_j}|^2 - |V^{j,r_j}_{r_j}|^2\right] \\
&\le C_L \left(1 + \sum_{j=1}^{\kappa-1} \mathbb{E}\left[|V^{j-1,r_j}_{r_j}|^2 - |V^{j,r_j}_{r_j}|^2\right]\right) \qquad (5.22)
\end{aligned}
$$

where the last inequality follows from (5.14).

4. For ease of notations, we now write $\mathbb{E}_{r_j}[\cdot]$ for $\mathbb{E}\left[\cdot \mid \mathcal{F}_{r_j}\right]$. By Cauchy-Schwartz inequality,

$$
\begin{aligned}
|V^{j-1,r_j}_{r_j}|^2 - |V^{j,r_j}_{r_j}|^2 &\le |V^{j-1,r_j}_{r_j} - V^{j,r_j}_{r_j}|\, |V^{j-1,r_j}_{r_j} + V^{j,r_j}_{r_j}| \\
&\le C_L\, \mathbb{E}_{r_j}[\beta]\, |V^{j-1,r_j}_{r_j} - V^{j,r_j}_{r_j}|\,, \qquad (5.23)
\end{aligned}
$$

where β is defined in Remark 5.6.

4.a. Recalling that ∇g, ∇h are bounded by L and that $\tau_{j-1} \le \tau_j \le T$, we observe that

$$
\begin{aligned}
& C_L \mathbf{1}_{\{\tau_{j-1}<\tau_j=T\}} + \left(\nabla h(X_{\tau_j})(\Lambda^t D_t X)_{\tau_j} - \nabla h(X_{\tau_{j-1}})(\Lambda^t D_t X)_{\tau_{j-1}}\right) \mathbf{1}_{\{\tau_{j-1}<T\}} \\
& \ge \nabla g(X_T) D_t X_T \mathbf{1}_{\{\tau_j=T\}} + \nabla h(X_{\tau_j})(\Lambda^t D_t X)_{\tau_j} \mathbf{1}_{\{\tau_j<T\}} \\
& -\nabla g(X_T) D_t X_T \mathbf{1}_{\{\tau_{j-1}=T\}} - \nabla h(X_{\tau_{j-1}})(\Lambda^t D_t X)_{\tau_{j-1}} \mathbf{1}_{\{\tau_{j-1}<T\}}\,.
\end{aligned}
$$

When (**H**1) holds, it then follows from (5.4), (5.8) and (5.17) that

$$
\begin{aligned}
|V^{j-1,r_j}_{r_j} - V^{j,r_j}_{r_j}| &\le C_L\, \mathbb{E}_{r_j}\left[\mathbf{1}_{\{\tau_{j-1}<\tau_j=T\}}\right] \\
&+ C_L \left(\mathbb{E}_{r_j}[\beta(\tau_j - \tau_{j-1})] + \bar\beta^{\frac12}\, \mathbb{E}_{r_j}[\beta(\tau_{j+1} - \tau_j)]^{\frac12}\right)\,.
\end{aligned}
$$

Since $\sum_{j=1}^{\kappa-1} \mathbf{1}_{\{\tau_{j-1}<\tau_j=T\}} \le 1$, the above inequality combined with (5.22) and (5.23) implies

$$
\begin{aligned}
\Sigma &\le C_L\, \mathbb{E}\left[1 + \sum_{j=1}^{\kappa-1} \mathbb{E}_{r_j}[\beta] \left(\mathbb{E}_{r_j}[\beta(\tau_j - \tau_{j-1})] + \bar\beta^{\frac12}\, \mathbb{E}_{r_j}[\beta(\tau_j - \tau_{j-1})]^{\frac12}\right)\right] \\
&\le C_L \left\{1 + \sum_{j=1}^{\kappa-1} \left(\mathbb{E}\left[\bar\beta\beta(\tau_j - \tau_{j-1})\right] + \mathbb{E}\left[\beta(\tau_j - \tau_{j-1})\right]^{\frac12}\right)\right\}
\end{aligned}
$$

where we used Cauchy-Schwartz inequality and (5.18). By (5.18) again, this shows that

$$
\begin{aligned}
\Sigma &\leq C_L \left\{1 + \mathbb{E}\left[\bar{\beta}\beta(\tau_{\kappa-1} - \tau_0)\right] + \sqrt{\kappa}\, \mathbb{E}\left[\beta(\tau_{\kappa-1} - \tau_0)\right]^{\frac{1}{2}}\right\} \\
&\leq C_L \left(1 + \sqrt{\kappa}\right) . \qquad (5.24)
\end{aligned}
$$

4.b. Under ($\mathbf{H}2'$), we use exactly the same arguments except that we appeal to (5.16) instead of (5.17). This leads to

$$
\Sigma \leq C_L \left\{1 + \sum_{j=1}^{\kappa-1} \mathbb{E}\left[\bar{\beta}\beta(\tau_j - \tau_{j-1})\right]\right\} \leq C_L . \qquad (5.25)
$$

5. By (5.19), (5.20), (5.21) and the definition of Σ in (5.22)

$$
\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|Z_t^b - Z_{t_i}^b|^2\right] \mathrm{d}t \leq C_L\, |\pi|\, (1 + \Sigma) .
$$

The proof is then concluded by appealing to (5.24), under ($\mathbf{H}1$), and to (5.25), under ($\mathbf{H}2'$), and by using Remark 3.5. □

Proof of Proposition 3.3 Let f_n be defined by :

$$
f_n(x, y, z) = \int_{\mathbb{R}^{2d+1}} \phi_n(x - \xi, y - \upsilon, z - \zeta) f(\xi, \upsilon, \zeta) \mathrm{d}\xi \mathrm{d}\upsilon \mathrm{d}\zeta ,
$$

with $\phi_n(x, y, z) = n^{2d+1}\phi(n(x, y, z))$ and ϕ a compactly supported smooth probability density function on $\mathbb{R}^{2d+1}$. Since f is L-lipschitz we have :

$$
||f - f_n||_\infty \leq \frac{C_L}{n} ,
$$

for some $C > 0$. Let σ_n, b_n, g_n, h_n be defined similarly for σ, b, g, h so that we have:

$$
||\sigma - \sigma_n||_\infty + ||b - b_n||_\infty + ||g - g_n||_\infty + ||h - h_n||_\infty \leq \frac{C_L}{n} .
$$

Let X^n be the forward diffusion associated to b^n and σ^n and let $(Y^{b,n}, Z^{b,n}, K^{b,n})$ be the solution of the discretely reflected BSDE (3.1) associated to X^n, f^n and g^n. Arguing as in Proposition 3.6 of [9], we get

$$
||Z^b - Z^{b,n}||_{\mathcal{H}^2}^2 \leq \frac{C_L}{n} . \qquad (5.26)
$$

Since, by Jensen's inequality,

$$
\begin{aligned}
||Z^b - \bar{Z}^b||_{\mathcal{H}^2} &\leq ||\bar{Z}^b - \bar{Z}^{b,n}||_{\mathcal{H}^2} + ||Z^b - Z^{b,n}||_{\mathcal{H}^2} + ||Z^{b,n} - \bar{Z}^{b,n}||_{\mathcal{H}^2} \\
&\leq 2\, ||Z^b - Z^{b,n}||_{\mathcal{H}^2} + ||Z^{b,n} - \bar{Z}^{b,n}||_{\mathcal{H}^2} ,
\end{aligned}
$$

the proof is concluded by applying Proposition 5.2 to $Z^{b,n}$, using (5.26) and letting n go to infinity. □

We now consider the case where the forward diffusion is approximated by its Euler scheme.

Proposition 5.3 *If* ($\mathbf{H}1$) *holds, then*

$$||Z^{b,e} - \bar{Z}^{b,e}||_{\mathcal{H}^2} \leq C_L \left(\kappa^{\frac{1}{4}} \, |\pi|^{\frac{1}{2}} + |\pi|^{\frac{1}{4}} \right) .$$

Proof. In view of Remark 5.2 and Remark 5.5, we can follow line by line the arguments of the proof of Proposition 5.2, after replacing the corresponding quantities in the definitions of β and $\bar{\beta}$, and re-defining, for $j \leq \kappa - 1$,

$$\begin{aligned} V_t^{j,s} &:= \mathbb{E}\left[\nabla g(X_T^\pi)(\Lambda^{e,s} D_s X^\pi)_T \mathbf{1}_{\{\tau_j^e = T\}} + \nabla h(X_{\tau_j^e})(\Lambda^{e,s} D_s X^\pi)_{\tau_j^e} \mathbf{1}_{\{\tau_j^e < T\}} \mid \mathcal{F}_t \right] \\ &+ \mathbb{E}\left[\int_s^{\tau_j^e} \nabla_x f(\Theta_u^{b,e})(\Lambda^{e,s} D_s X^\pi)_u \mathrm{d}u \mid \mathcal{F}_t \right] , \; s \leq t . \end{aligned} \tag{5.27}$$

The only difference appears in step 2. Instead of using a relation like (5.3) for X^π (which does not hold), we use the martingale property of V^{j,t_i} on $[t_i, t_{i+1})$ and write

$$\begin{aligned} \mathbb{E}\left[|V_t^{j,t_i} - V_{t_i}^{j,t_i}|^2 \right] &\leq \mathbb{E}\left[|V_{t_{i+1}}^{j,t_i}|^2 - |V_{t_i}^{j,t_i}|^2 \right] \\ &\leq \mathbb{E}\left[|V_{t_{i+1}}^{j,t_{i+1}}|^2 - |V_{t_i}^{j,t_i}|^2 + |V_{t_{i+1}}^{j,t_{i+1}} - V_{t_{i+1}}^{j,t_i}| \, |V_{t_{i+1}}^{j,t_{i+1}} + V_{t_{i+1}}^{j,t_i}| \right] , \end{aligned}$$

where by (5.5), (5.11), (5.13) and Cauchy-Schwartz inequality

$$\mathbb{E}\left[|V_{t_{i+1}}^{j,t_{i+1}} - V_{t_{i+1}}^{j,t_i}| \, |V_{t_{i+1}}^{j,t_{i+1}} + V_{t_{i+1}}^{j,t_i}| \right] \leq C_L \sqrt{|\pi|} .$$

The inequality (5.21) then becomes

$$\mathbb{E}\left[|V_t^{j,t_i} - V_{t_i}^{j,t_i}|^2 \right] \leq \mathbb{E}\left[|V_{t_{i+1}}^{j,t_{i+1}}|^2 - |V_{t_i}^{j,t_i}|^2 \right] + C_L \sqrt{|\pi|} .$$

□

Proof of Proposition 3.4 The required result follows from Proposition 5.3 and by arguing as in the proof of Proposition 3.3. □

We conclude this section with the proof of Proposition 3.1.

Proof of Proposition 3.1. Assume that ($\mathbf{H}'$) holds. By Remark 5.4, we have

$$\mathbb{E}\left[\int_{t_i}^{t_{i+1}} |Z_s^b|^2 ds \right] \leq C_L \, |\pi| .$$

Arguing as in the proof of Proposition 3.3, we obtain that the above bound holds without ($\mathbf{H}'$). The required result then follows from Itô's Lemma, the Lipschitz-continuity of f, (2.2), the bound on Y^b given in (3.3) and Burkholder-Davis-Gundy's inequality, recall (3.1). □

Appendix

Proof of Proposition 3.2

1. Set $\delta Y = Y^b - Y^\pi$, $\delta\tilde{Y} = \tilde{Y}^b - \tilde{Y}^\pi$, $\delta Z = Z^b - Z^\pi$, $\delta f_s = f(X_s, \tilde{Y}^b_s, Z^b_s) - f(X^\pi_{t_i}, \tilde{Y}^\pi_{t_i}, \bar{Z}^\pi_{t_i})$ for $s \in [t_i, t_{i+1})$. Recalling (3.2), (3.6), (3.7), the fact that $\Re \subset \pi$ and using Itô's Lemma, we compute that for $t \in [t_i, t_{i+1})$

$$A^i_t \;:=\; \mathbb{E}_i\left[|\delta\tilde{Y}_t|^2 + \int_t^{t_{i+1}} |\delta Z_s|^2 \mathrm{d}s - \left|\delta Y_{t_{i+1}}\right|^2\right] = \mathbb{E}_i\left[\int_t^{t_{i+1}} 2\delta\tilde{Y}_s \delta f_s \mathrm{d}s\right] \,,$$

recall that $\mathbb{E}_i[\cdot]$ stands for $\mathbb{E}[\cdot \mid \mathcal{F}_{t_i}]$. By (3.10), the Lipschitz-continuity of f and the inequality $xy \le cx^2 + c^{-1}y^2$, for $x, y \in \mathbb{R}_+$ and $c > 0$, we therefore obtain

$$\begin{aligned} A^i_t \;&\le\; \mathbb{E}_i\left[\int_t^{t_{i+1}} \alpha|\delta\tilde{Y}_s|^2 \mathrm{d}s + \frac{C_L}{\alpha}\left(|\pi|\; |\delta\tilde{Y}_{t_i}|^2 + \int_{t_i}^{t_{i+1}} |\delta Z_s|^2 \mathrm{d}s\right)\right] \\ &+\; \frac{C_L}{\alpha}\mathbb{E}_i\left[\int_t^{t_{i+1}} |X_s - X^\pi_{t_i}|^2 + |\tilde{Y}^b_s - \tilde{Y}^b_{t_i}|^2 + |Z^b_s - \bar{Z}^b_{t_i}|^2 \mathrm{d}s\right] \end{aligned}$$

where α is a positive parameter to be chosen later on. Using Gronwall's Lemma and taking α large enough, we deduce that, for $|\pi|$ small enough, there is some $\eta > 0$, independent of π, such that

$$\mathbb{E}_i\left[|\delta\tilde{Y}_{t_i}|^2 + \eta\int_{t_i}^{t_{i+1}} |\delta Z_s|^2 \mathrm{d}s\right] \;\le\; e^{C_L|\pi|}\mathbb{E}_i\left[|\delta Y_{t_{i+1}}|^2\right] + C_L B_i \tag{5.28}$$

$$\sup_{t\in[t_i,t_{i+1}]} \mathbb{E}_i\left[|\delta\tilde{Y}_t|^2\right] \;\le\; C_L\left(\mathbb{E}_i\left[|\delta Y_{t_{i+1}}|^2\right] + |\pi|\; |\delta\tilde{Y}_{t_i}|^2 + B_i\right) \tag{5.29}$$

where

$$B_i \;:=\; \mathbb{E}_i\left[\int_{t_i}^{t_{i+1}} |X_s - X^\pi_{t_i}|^2 + |\tilde{Y}^b_s - \tilde{Y}^b_{t_i}|^2 + |Z^b_s - \bar{Z}^b_{t_i}|^2 \mathrm{d}s\right] \,.$$

2. Since $|\delta Y_{t_i}| \le \max\{|\delta\tilde{Y}_{t_i}|; |h(X_{t_i}) - h(X^\pi_{t_i})|\mathbf{1}_{t_i\in\Re}\}$ for $i < N$, see (3.1), (3.4) and (3.7), it follows from (5.28) applied at $t = t_i$ and the Lipschitz-continuity of h that, for $|\pi|$ small enough,

$$|\delta Y_{t_i}| \;\le\; \max\left\{e^{C_L|\pi|}\mathbb{E}_i\left[|\delta Y_{t_{i+1}}|^2\right] + C_L B_i \;;\; L|X_{t_i} - X^\pi_{t_i}|\mathbf{1}_{t_i\in\Re}\right\} \,. \tag{5.30}$$

Since $|\delta Y_{t_N}| \le L|X_{t_N} - X^\pi_{t_N}|$, by the Lipschitz-continuity of g, we deduce from (2.5), (5.30) and an inductive argument that

$$\max_{i\le N} \mathbb{E}\left[|\delta Y_{t_i}|^2\right] \;\le\; C_L\;\left(N|\pi|^2 + \bar{B}\right)$$

with

$$\bar{B} := \mathbb{E}\left[\sum_{i=0}^{N-1} B_i\right] \,.$$

Since by assumption $N|\pi| \leq L$, this implies

$$\max_{i \leq N} \mathbb{E}\left[|\delta Y_{t_i}|^2\right] \quad \leq \quad C_L \ \left(|\pi| + \bar{B}\right) \ . \tag{5.31}$$

3. Observing that for $s \in [t_i, t_{i+1})$

$$\mathbb{E}\left[\left|\tilde{Y}_s^b - \tilde{Y}_{t_i}^b\right|^2\right] \quad \leq \quad C_L \int_{t_i}^{s} \mathbb{E}\left[|f(\Theta_u^b)|^2 + |Z_u^b|^2\right] \mathrm{d}u$$

it follows from (2.2), (3.3), the Lipschitz-continuity of f and the assumption $N|\pi| \leq L$ that

$$\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\tilde{Y}_s^b - \tilde{Y}_{t_i}^b|^2\right] ds \quad \leq \quad C_L \ |\pi| \ .$$

Combined with (2.4), this implies

$$\bar{B} \quad \leq \quad C_L \ \left(|\pi| + ||Z^b - \bar{Z}^b||^2_{\mathcal{H}^2}\right) \ . \tag{5.32}$$

In view of (5.28) and (5.31), this leads to

$$\begin{aligned} \mathbb{E}\left[|\delta\tilde{Y}_{t_i}|^2 + \eta \int_{t_i}^{t_{i+1}} |\delta Z_s|^2 \mathrm{d}s\right] \quad &\leq \quad (1 + C_L \ |\pi|) \ \mathbb{E}\left[|\delta Y_{t_{i+1}}|^2 + C_L B_i\right] \ , \\ &\leq \quad C_L \left(|\pi| + ||Z^b - \bar{Z}^b||^2_{\mathcal{H}^2}\right) \end{aligned} \tag{5.33}$$

which, by (3.1), (3.7), (5.29), (5.31) and (5.32) shows that

$$\sup_{t \leq T} \mathbb{E}\left[|\delta Y_t|^2\right] + \sup_{t \leq T} \mathbb{E}\left[|\delta\tilde{Y}_t|^2\right] \quad \leq \quad C_L(|\pi| + ||Z^b - \bar{Z}^b||^2_{\mathcal{H}^2}) \ . \tag{5.34}$$

Let i_j be defined through $t_{i_j} = r_j$. Using (3.1) and (3.7) again, we deduce from (5.33) and (5.34) that

$$\begin{aligned} \mathbb{E}\left[\int_{r_j}^{r_{j+1}} |\delta Z_s|^2 \mathrm{d}s\right] \quad = \quad \sum_{k=i_j}^{i_{j+1}-1} \mathbb{E}\left[\int_{t_k}^{t_{k+1}} |\delta Z_s|^2 \mathrm{d}s\right] \quad &\leq \quad C_L \ \left(\mathbb{E}\left[|\delta Y_{r_{j+1}}|^2\right] + \sum_{k=i_j}^{i_{j+1}-1} B_k\right) \\ &\leq \quad C_L \ \left(|\pi| + \sum_{k=i_j}^{i_{j+1}-1} B_k\right) \end{aligned} \tag{5.35}$$

so that, by (5.32),

$$||Z^b - Z^\pi||^2_{\mathcal{H}^2} \quad = \quad \mathbb{E}\left[\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} |\delta Z_s|^2 \mathrm{d}s\right] \quad \leq \quad C_L \left(\kappa \ |\pi| + ||Z^b - \bar{Z}^b||^2_{\mathcal{H}^2}\right) \ .$$

This proves the second claim of Proposition 3.2.

4. Using Burkholder-Davis-Gundy's inequality and arguing as in the first steps of 1, we now compute that

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[r_j,r_{j+1}]}|\delta\tilde{Y}_t|^2\right] &\leq \mathbb{E}\left[\sup_{t\in[r_j,r_{j+1})}|\delta\tilde{Y}_t|^2+|\delta\tilde{Y}_{r_{j+1}}|^2\right] \\
&\leq C_L\mathbb{E}\left[|\delta Y_{r_{j+1}}|^2+\int_{r_j}^{r_{j+1}}\left(|\delta f_s|^2+|\delta Z_s|^2\right)\mathrm{d}s+|\delta\tilde{Y}_{r_{j+1}}|^2\right] \\
&\leq C_L\left(\bar{B}+\mathbb{E}\left[|\delta Y_{r_{j+1}}|^2+\int_{r_j}^{r_{j+1}}|\delta Z_s|^2\mathrm{d}s\right]+\max_{i\leq N}\mathbb{E}\left[|\delta\tilde{Y}_{t_i}|^2\right]\right) \\
&\leq C_L\left(|\pi|+||Z^b-\bar{Z}^b||^2_{\mathcal{H}^2}\right)
\end{aligned}$$

where we used (5.32), (5.34) and (5.35). Since

$$|\delta Y_t| \leq |\delta\tilde{Y}_t|+|h(X_t)-h(X_t^\pi)|$$

the first assertion of Proposition 3.2 follows from the Lipschitz-continuity of h and (2.4).

5. In the case (PS) where $X^\pi=X$ on π, we argue exactly as above up to (5.33). In this case, we have $|\delta Y_t|\leq|\delta\tilde{Y}_t|$ for all $t\in\pi$. Thus replacing $\delta\tilde{Y}_{t_i}$ by δY_{t_i} in (5.33) and summing up over i implies

$$\mathbb{E}\left[\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}|\delta Z_s|^2\mathrm{d}s\right]\leq C_L\left(\max_{t\in\Re}\mathbb{E}\left[|\delta Y_t|^2\right]+\bar{B}\right).$$

The proof is then concluded as above. □

Proof of Proposition 4.1.

For seek of completeness, we give here the proof of Proposition 4.1. We only consider the error on (Y,Z), the error on (Y^e,Z^e) is treated similarly.

1. We assume that $(\mathbf{H}3)$ and $(\mathbf{H}')$ hold. Let us define

$$\delta Y=Y-Y^b\,,\delta\tilde{Y}=Y-\tilde{Y}^b\,,\delta Z=Z-Z^b\,,\delta f=f(X,Y,Z)-f(X,\tilde{Y}^b,Z^b)\,,$$

and remark that

$$|\delta Y_t| = |\delta\tilde{Y}_t|\,,\forall t\notin\Re \text{ and } |\delta Y_t| \leq |\delta\tilde{Y}_t|\,,\forall t\in\Re\,. \tag{5.36}$$

Applying Itô formula to $|\delta\tilde{Y}|^2$, we obtain for $t\in[r_j,r_{j+1})$

$$\mathbb{E}\left[|\delta\tilde{Y}_t|^2+\int_t^{r_{j+1}}|\delta Z_s|^2\mathrm{d}s\right] = \mathbb{E}\left[|\delta Y_{r_{j+1}}|^2+2\int_t^{r_{j+1}}\delta\tilde{Y}_s\delta f_s\mathrm{d}s+2\int_t^{r_{j+1}}\delta\tilde{Y}_s\mathrm{d}K_s\right]. \tag{5.37}$$

Following the arguments of 1. in the proof of Proposition 3.2 and using (5.36), we obtain that, for some $\eta > 0$ depending only on L,

$$\mathbb{E}\left[|\delta\tilde{Y}_t|^2 + \eta\int_t^{r_{j+1}} |\delta Z_s|^2 \mathrm{d}s\right] \;\le\; (1 + C_L|\Re|)\,\mathbb{E}\left[|\delta\tilde{Y}_{r_{j+1}}|^2 + 2\int_t^{r_{j+1}} \delta\tilde{Y}_s \mathrm{d}K_s\right] . \tag{5.38}$$

Now observe that, by (4.2),

$$\int_t^{r_{j+1}} \delta\tilde{Y}_s \mathrm{d}K_s \;=\; \int_t^{r_{j+1}} (h(X_s) - \tilde{Y}^b_s)\mathrm{d}K_s \;, \tag{5.39}$$

where

$$\begin{aligned} h(X_s) - \tilde{Y}^b_s \;&=\; \mathbb{E}\left[h(X_s) - Y^b_{r_{j+1}} - \int_s^{r_{j+1}} f(X_u, \tilde{Y}^b_u, Z^b_u)\mathrm{d}u \mid \mathcal{F}_s\right] \\ &\le\; \mathbb{E}\left[h(X_s) - h(X_{r_{j+1}}) + \int_s^{r_{j+1}} |f(X_u, \tilde{Y}^b_u, Z^b_u)|\mathrm{d}u \mid \mathcal{F}_s\right] . \end{aligned} \tag{5.40}$$

Set

$$\begin{aligned} \xi \;&:=\; \sup_{t\le T}\left(1 + |\tilde{Y}^b_t| + |X_t| + |\rho_1(X_t)| + |\rho_2(X_t)| + |X_t|^2 + |Z^b_t|\right)^2 \\ \bar{\xi} \;&:=\; \sup_{s\le T}\mathbb{E}\left[\xi \mid \mathcal{F}_s\right] \;, \end{aligned}$$

which, by (2.2), (3.3) and Remark 5.4, satisfy

$$||\xi||_{\mathcal{S}^p} + ||\bar{\xi}||_{\mathcal{S}^p} \;\le\; C^p_L \,. \tag{5.41}$$

Using (**H**3) and the Lipschitz-continuity of b, σ and f, (5.40) becomes

$$\begin{aligned} h(X_s) - \tilde{Y}^b_s \;&\le\; C_L\mathbb{E}\left[\int_s^{r_{j+1}} (1 + |\rho_1(X_s)'b(X_u)| + |\rho_2(X_s)|(1 + |X_u|^2))\mathrm{d}u \mid \mathcal{F}_s\right] \\ &\quad + C_L\,\mathbb{E}\left[\int_s^{r_{j+1}} (|X_u| + |\tilde{Y}^b_u| + |Z^b_u|)\mathrm{d}u \mid \mathcal{F}_s\right] \\ &\le\; C_L\,|\Re|\,\bar{\xi}\,. \end{aligned}$$

Combining the last inequality with (5.39), it follows that

$$\mathbb{E}\left[\int_t^{r_{j+1}} \delta\tilde{Y}_s \mathrm{d}K_s\right] \;\le\; C_L|\Re|\mathbb{E}\left[\bar{\xi}\,(K_{r_{j+1}} - K_t)\right] \;, \tag{5.42}$$

which by (5.38) leads to

$$\begin{aligned} \mathbb{E}\left[|\delta\tilde{Y}_{r_j}|^2 + \eta\int_{r_j}^{r_{j+1}} |\delta Z_s|^2 \mathrm{d}s\right] \;&\le\; (1 + C_L|\Re|)\left(\mathbb{E}\left[|\delta\tilde{Y}_{r_{j+1}}|^2\right]\right. \\ &\qquad \left. + C_L|\Re|\,\mathbb{E}\left[\bar{\xi}\,(K_{r_{j+1}} - K_{r_j})\right]\right) . \end{aligned}$$

This shows that

$$\mathbb{E}\left[|\delta\tilde{Y}_{r_j}|^2\right] \quad \leq \quad C_L|\Re| \; . \tag{5.43}$$

Summing up in the previous inequality and using (5.43), we get

$$||\delta Z||^2_{\mathcal{H}^2} \quad \leq \quad C_L|\Re| \; . \tag{5.44}$$

In view of (5.36), (5.38) and (5.42), (5.43) implies

$$\sup_{t\in[0,T]} \|\delta Y_t\|^2_{L^2} \;\leq\; \sup_{t\in[0,T]} \|\delta\tilde{Y}_t\|^2_{L^2} \leq C_L\;|\Re| \; . \tag{5.45}$$

2. We now assume that $(\mathbf{H}1')$: h is C^2_b with first and second derivatives bounded by L. We argue as in 2 of the proof of Proposition 3.2. Recalling (5.36), we first compute that, for $j \leq \kappa - 1$,

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[r_j,r_{j+1}]} |\delta\tilde{Y}_t|^2\right] \quad &\leq \quad C_L\mathbb{E}\left[|\delta\tilde{Y}_{r_{j+1}}|^2 + \int_{r_j}^{r_{j+1}} \left(|\delta f_s|^2 + |\delta Z_s|^2\right) \mathrm{d}s\right] \\ &+ \quad C_L\mathbb{E}\left[(K_{r_{j+1}} - K_{r_j})^2\right] \; .\end{aligned} \tag{5.46}$$

By Proposition 4.2 in [9], it follows from $(\mathbf{H}1')$ that $0 \leq \mathrm{d}K_t \leq k_t\mathrm{d}t$, in the sense of measures, for some adapted process k satisfying $||k||_{\mathcal{S}^p} \leq C^p_L$. This implies that

$$\mathbb{E}\left[\sup_{t\in[r_j,r_{j+1}]} |\delta\tilde{Y}_t|^2\right] \quad \leq \quad C_L\mathbb{E}\left[|\delta\tilde{Y}_{r_{j+1}}|^2 + \int_{r_j}^{r_{j+1}} \left(|\delta f_s|^2 + |\delta Z_s|^2\right) \mathrm{d}s\right] + C_L\;|\Re| \; .$$

Using (2.2), (3.3), (5.36), (5.45) and (5.44), this shows that

$$\mathbb{E}\left[\sup_{t\in[r_j,r_{j+1}]} |\delta Y_t|^2\right] \leq \mathbb{E}\left[\sup_{t\in[r_j,r_{j+1}]} |\delta\tilde{Y}_t|^2\right] \quad \leq \quad C_L\;|\Re| \; .$$

3. To conclude the proof, we have to remove the assumption $(\mathbf{H}')$ for the first assertion and $(\mathbf{H}1')$ for the second one. This is done by using the same approximation arguments as in the proof of Proposition 3.3. □

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